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Exact Height Probabilities in the Abelian Sandpile Model

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Abstract. We study Bak, Tang and Wiesenfeld's Abelian sandpile model of self-organized criticality on 2D square lattice. A combinatorical method for evaluation of height probabilities is proposed. Exact analytical expression for the fractional number of sites having height 2 is obtained.

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Sandpile models originally proposed by Bak, Tang, and Wiesenfeld [1] attract now a lot of attention as the simplest models that capture essential properties of the self-organized critical state (SOC). Recently, Dhar [2] has shown that the sandpile automaton model has an Abelian group structure which permitted him to find the total number of allowed configurations in the SOC state. Also, he found the correlation function measuring the expected number of topplings at a given site due to a particle added at another one.

Seeking a more direct characterization of the SOC state, Majumdar and Dhar [3] determined $P(1)$, the fractional number of sites having height 1 and $P_{11}(r)$, the probability that two sites separated by a distance r both have height 1. However, the problem of finding the other height probabilities and correlations between them turned out more difficult. So far, these quantities have been calculated analytically only for the Bethe lattice [4]. The first numerical estimations of $P(2), P(3), P(4)$ for the square lattice were made by Zhang [5] for a model with continuous heights: $P(2) = 0.16; P(3) = 0.32; P(4) = 0.42$. The related data for the Abelian sandpile model on the lattice of linear sizes $30, 40$ were obtained by Erzan and Sinha [6]: $P(2) = 0.17 \pm 7\%$; $P(3) = 0.31 \pm 9\%$; $P(4) = 0.45 \pm 3\%$. The most accurate calculations for a lattice size 672 were undertaken by Manna[7] who found $P(2) = 0.174$; $P(3) = 0.307$; $P(4) = 0.446$ with typical errors of an order of 0.003. Attempts of analytical determination $P(2)$ showed a very slow convergence of cluster series [3] and gave only the lower bound $P(2) \geq 0.131438$.

In this letter, I present a method leading to exact solution of the problem in two dimensions. In particular, I give an analytical formula for $P(2)$ that reads in the limit of an infinitely large lattice:

$$P(2) = \frac{1}{2} - \frac{3}{2\pi} - \frac{2}{\pi^2} + \frac{12}{\pi^3} + \frac{I_0}{4} \tag{1}$$

with

$$I_0 = \frac{1}{(2\pi)^4} \iiiint_0^{2\pi} \frac{i \sin(\beta_1) \det(M)}{D(\alpha_1, \beta_1) D(\alpha_2, \beta_2) D(\alpha_1 + \alpha_2, \beta_1 + \beta_2)} d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 \tag{2}$$

where

$$D(\alpha, \beta) = 2 - \cos(\alpha) - \cos(\beta) \tag{3}$$

and M is a 4×4 matrix

$$M = \begin{pmatrix} 1 & 1 & e^{i\alpha_2} & 1 \\ 3 & e^{i(\beta_1+\beta_2)} & e^{i(\alpha_2-\beta_2)} & e^{i\beta_1} \\ \frac{4}{\pi} - 1 & e^{i(\alpha_1+\alpha_2)} & 1 & e^{-i\alpha_1} \\ \frac{4}{\pi} - 1 & e^{-i(\alpha_1+\alpha_2)} & e^{2i\alpha_2} & e^{i\alpha_1} \end{pmatrix} \tag{4}$$

The numerical evaluation of the integral (2) leads to $P(2) = 0.1739\ldots$. The solution is based on an analogy between configurations of sandpiles and spanning trees, i.e., tree-like graphs covering all sites of a given lattice.

We start with recalling the definition of the model. Consider a large square lattice L consisting of n sites. The sandpile is characterized by integer heights z_i

at all sites i and is specified by two rules. (i) Adding a particle at a random site: $z_i \to z_i + 1$; (ii) The toppling rule: if any $z_i > 4$, then $z_i \to z_i - 4$ and $z_j \to z_j + 1$, $|i - j| = 1$. In a stable configuration, the height z_i at any site i takes values 1,2,3,4.

Following Dhar [2], we define a forbidden subconfiguration (FSC) as any subset $F \subset L$ of sites if the corresponding heights $\{z_j\}$, $j \in F$, satisfy the inequalities: $z_j \leq$ coordination number of j in F. A configuration that contains no FSCs is called an allowed configuration.

Dhar proposed a recursive procedure called the burning algorithm to determine if a given configuration is allowed. One deletes step by step from a given configuration any site j whose height z_j is greater than the coordination number of j in a lattice resulting after the preceding step. If in the end the lattice becomes empty, the configuration is allowed. The number of stable allowed configurations is given by the remarkable simple formula [2]:

$$N = det\Delta \tag{5}$$

where Δ is an $n \times n$ discrete Laplacian matrix with $\Delta_{ij} = 4$ if $i = j$; $\Delta_{ij} = -1$ if $|i - j| = 1$, $\Delta_{ij} = 0$ otherwise.

For a given lattice site i_0, the set of allowed configurations can be divided into four subsets s_1, s_2, s_3, s_4. These are defined as follows. A configuration C belongs: to a subset s_1 if it remains allowed after all substitutions $z_0 = 1, 2, 3, 4$ at i_0; to a subset s_2 if it remains allowed for $z_0 = 2, 3, 4$ and becomes forbidden for $z_0 = 1$; to subset s_3 if it remains allowed for $z_0 = 3, 4$ and becomes forbidden for $z_0 = 1, 2$. The subset s_4 contains configurations which are allowed only for $z_0 = 4$. The height probabilities $P(1), P(2), P(3), P(4)$ now can be written in the form:

$$P(1) = \frac{N_1}{4N}; P(2) = P(1) + \frac{N_2}{3N}; P(3) = P(2) + \frac{N_3}{2N}; P(4) = P(3) + \frac{N_4}{N} \tag{6}$$

where N_i is the number of allowed configurations in the subset s_i, $i = 1, ..., 4$. The description of the subset s_1 is given by Majumdar and Dhar [3] who obtained $P(1) = 2/\pi^2 - 4/\pi^3$.

Let us consider the subset s_2. Denote the four neighbor sites of i_0 by j_1, j_2, j_3, j_4 numbered in clockwise order. By definition, the substitution $z_0 = 1$ converts an arbitrary configuration $C \in s_2$ into a forbidden one C'. It means that FSC appears which contains the site i_0 with $z_0 = 1$, one of the sites $j_1, ..., j_4$, say j_1, with $z_{j_1} \geq 1$ and some k connected sites($k \geq 0$) *including none of the sites* j_2, j_3, j_4. (If one of j_2, j_3, j_4 also belongs to FSC, then the configuration C' remains forbidden after the substitution $z_0 = 2$).

Let $S(C)$ be the FSC resulting from the substitution $z_0 = 1$ in C. We construct a lattice L' in the following way. We delete the boundary bonds connecting the sites in $S(C)$ to the rest of the lattice L with the exception of the only bond connecting the site i_0 with one of the sites j_2, j_3, j_4 (j_2 for definiteness). For each bond deleted, we also decrease the maximum height allowed at the two end sites of the bond by 1. In this way, we obtain a new toppling rule matrix $\Delta'(S)$ which depends on the form of a given FSC. Due to coincidence burning procedures, the set of all allowed configurations on the lattice L' is in one-to-one correspondence to the set of configurations C which generate S by the substitution $z_0 = 1$. As the sites $j_1, ..., j_4$ are equivalent and three possibilities $z_0 = 2, 3, 4$ contribute to s_2, the number of allowed configurations in s_2 is

$$N_2 = 12 \sum_S det\Delta'(S) \tag{7}$$

where the sum runs over all possible FSCs containing the sites i_0, j_1 and none of the sites j_2, j_3, j_4.

Let us now look at (5) and (7) from a different point of view. To further simplify the problem, we specify the boundary conditions as follows: $\Delta_{ii} = 3$ if i belongs to the edge of L, $\Delta_{ii} = 2$ if i belongs to one of three corners and $\Delta_{ii} = 3$ if i coincides with the fourth corner denoted by $\star$.

Definition A subgraph G of L is a subset of vertices and bonds of L such that it forms a graph. Denote by $\nu(G)$, $\mu(G)$ and $\kappa(G)$ the numbers of vertices, connected parts and internal loops of G. A subgraph T is a spanning tree of L if $\nu(T) = \nu(L)$, $\mu(T) = 1$ and $\kappa(T) = 0$.

According to the Kirchhoff theorem [8], $det\Delta$ is the number of spanning trees of the lattice L. By construction of $\Delta'(S)$, the sum $\sum det\Delta'(S)$ is the number of spanning trees T' satisfying the following conditions:

(a) Each T' contains the bonds i_0j_1 and i_0j_2;

(b) Deletion of the bond i_0j_2 divides T' into two connected subtrees T_1 and T_2 such that the sites i_0 and j_1 belong to T_1 and the sites $\star, j_2, j_3, j_4$ belong to T_2.

(c) The bonds i_0j_3 and i_0j_4 are always absent among the bonds of T'.

It is convenient to introduce a different description of tree configurations. Let each lattice site i except $\star$ contain an arrow which can be directed from i to one of its nearest neighbors i'. We say that an arrow generates a path ii' from i to i'. A collection of path of the form i_1i_2, i_2i_3, ... , $i_{k-1}i_k$ is a path i_1i_k from i_1 to i_k. If the site i_k coincides with i_1, the path i_1i_k is closed.

The configurations of arrows generating no closed paths are in one-to-one correspondence to the spanning trees of the given lattice. Indeed, let us ascribe to each vertex i of the tree an arrow directed from i to the nearest neighbor i' for which a distance (the number of connected bonds) between i' and $\star$ is minimal. We get a configuration of arrows which generates no closed paths. Conversely, consider an arrow configuration. The absence of closed paths implies that each generated path ends at the site $\star$. Then a collection of bonds belonging to all paths forms a spanning tree having the root $\star$.

Now, we can reformulate the rules (a),(b),(c) in the arrow language. It follows from (a) and (b) that the arrow at i_0 is directed to j_2 and the arrow at j_1 to i_0. The condition (c) implies that arrows at j_3 and j_4 are directed anywhere but not to i_0. The condition (b) implies also that all paths starting at the sites of T_1 pass to $\star$ via j_1. On the contrary, there are no paths from the sites of T_2 to j_1 (and consequently from j_4 to j_1). To fulfill the latter condition, we put *one more arrow* at i_0 directed to j_4 and demand that the new configuration of arrows is also acyclic, i.e., it does not generate any closed path. The resulting combination of arrows at i_0, j_1, j_2, j_3, j_4 denoted by C_0 is shown in Fig.1. Our problem, therefore, is reduced to finding

$N(C_0)$, the number of acyclic configurations of arrows containing C_0. Taking into account (6) and (7) we get the following intermediate result

$$P(2) - P(1) = \frac{4N(C_0)}{N} \tag{8}$$

Enumeration of trees or arrows configurations obeying the formulated rules comes out of validity of the Kirchhoff theorem. To introduce the necessary improvements, we shall consider a combinatorial content of this theorem.

Let $\Delta(x,y)$ be a $n \times n$ matrix with elements $\Delta_{ij}(x,y) = y$ if $i = j$, $\Delta_{ij}(x,y) = -x$ if $|i-j| = 1$, $\Delta_{ij}(x,y) = 0$ otherwise. It is easy to show [9] that the function

$$g(x,y) = det\Delta(x,y) \tag{9}$$

is the generating function of all possible configurations of closed paths each bond of which has the weight x and each path brings a minus sign. The paths have no self-intersections and no two paths have a common lattice site. Sites not belonging to any path have the weight y. At $y = 4$ and $x = 1$ (9) coincides with (5) and works as the well known inclusion-exclusion principle [10]: in the expansion of determinant, diagonal elements of $\Delta(x,y)$ generate all possible placements of arrows and non-diagonal ones exclude those generating closed paths.

If a given site contains two fixed arrows, action of the inclusion-exclusion principle becomes more complicated. In contrast with the standard acyclic situation, configurations of arrows may appear which generate two closed paths having common sites: a path P_1 of type $i_0 j_2 ... j_1 i_0$ and a path P_2 of type $i_0 j_4 ... j_1 i_0$ (Fig.1). So, our task consists of two parts. First, we should provide cancellation both of P_1 and P_2. Second, as configurations containing P_1 and P_2 simultaneously will be excluded twice (due to P_1 and P_2), we must return these into the expansion.

The first problem is relatively simple. We introduce two matrices $\Delta^{(1)} = \Delta + \delta_{(1)}$ and $\Delta^{(2)} = \Delta + \delta_{(2)}$. The defect matrix $\delta_{(1)}$ should be such that the following matrix elements $[i,j]$ of $\Delta^{(1)}_{ij}$ equal zero: $[i_0, j']$ where j' is any nearest neighbor site of i_0 except j_2; $[j_1, j'']$ where j'' is any n.n. site of j_1 except i_0, and also elements $[j_3, i_0]$ and $[j_4, i_0]$. The matrix $\delta_{(2)}$ converts to zero the following elements of $\Delta^{(2)}_{ij}$: $[i_0, j']$ where j' is any n.n. site of i_0 except j_4; $[j_1, j'']$ where j'' is any n.n. site of j_1 except i_0, and elements $[j_2, i_0]$ and $[j_3, i_0]$. In addition, the matrix element $[j_1, i_0]$ becomes $-\epsilon$.

Then, according to the Kirchhoff theorem, $det\Delta^{(1)}$ enumerates all possible configurations of arrows containing the subconfiguration C_0 except the arrow directed from i_0 to j_4 and generating no closed paths including P_1. The other expression, $\lim[det\Delta^{(2)}/\epsilon]$ as $\epsilon \to \infty$ gives all configurations containing C_0 except the arrow directed from i_0 to j_2 and generating *precisely one closed path* of type P_2 weighted with minus sign. The sum of these determinants gives configurations which contain C_0, generate neither P_1 nor P_2 separately and, possibly, generate a combination of P_1 and P_2 having a form of a Θ-graph (Fig.1). Each Θ-graph being excluded twice brings a minus sign.

The second problem consists in enumeration of arrows configurations generating a Θ-graph. This is a crucial point of the solution. Let us first describe the Θ-graph more explicitly. For a given site i of a subgraph $G \subset L$, denote by $deg(i)$ the number of its neighboring sites $j \in G$ for which the bond ij also belongs to G. A Θ-graph is a subgraph of L containing the sites of two types: sites j with $deg(j) = 2$ and two sites, i_0 and i_1, with $deg(i_0) = deg(i_1) = 3$. For a Θ-graph in Fig.1 the site i_0 is surrounded by three sites j_1, j_2, j_4 and the site i_1 by the sites a, b, c. The second group of sites may be oriented arbitrary with respect to the first one.

We can try to construct a Θ-graph as follows. For fixed positions of the point i_1 and its neighbors a, b, c we should define a generating function of arrow configurations which generate three paths π_1, π_2, π_3 starting at sites a, b, c and ending at sites j_2, j_0, j_4. The combination of paths π_1, π_2, π_3 is equivalent to a Θ-graph (with inverted arrows on the bonds belonging to two of them). But a generating function of type (9) generates only closed paths having no endpoints. To overcome this difficulty, we add to the original square lattice L three "bridges", additional bonds connecting the sites a and j_2, c and j_4, b and i_0. Accordingly, we introduce the matrix $\Delta^{(3)} = \Delta + \delta_{(3)}$ with a defect matrix $\delta_{(3)}$ such that the three new nonzero elements of $\Delta^{(3)}$ appear: $[j_2, a] = [j_4, c] = [i_0, b] = -\epsilon$. As above, the element $[j_3, i_0]$ becomes zero. Also, $\delta_{(3)}$ converts to zero the elements $[i_1, j']$ where j' is any n.n. site of i_1 except b. Then, applying the formula (9) to the new lattice $\tilde{L}$, we conclude that the expression $\lim[det\Delta^{(3)}/\epsilon^3]$ as $\epsilon \to \infty$ gives all possible configurations of arrows on $\tilde{L}$ generating *either three closed paths* of type $j_2 a ... j_2$; $i_0 b ... j_1 i_0$; $j_4 c ... j_4$ or *a single closed path* of type $j_2 a ... j_1 i_0 b ... j_4 c ... j_2$ or of type $j_2 a ... j_4 c ... j_1 i_0 b ... j_2$. In both cases the arrows of closed paths belonging to the lattice L form the paths π_1, π_2, π_3 and therefore the desirable Θ-graph (with minus sign). Summation over all possible positions of the site i_1 and its three n.n. gives the necessary improvement of the inclusion-exclusion expansion.

Remark It is easy to check that the appearance of two closed paths passing via three bridges is forbidden in the 2D case for topological reasons. It is not the case for the 3D lattice. As the control of sign is impossible in the presence of both even and odd numbers of closed paths, our solution is restricted to the 2D case.

Practically, however, it is more convenient to use three different matrices $\Delta_{i_1}(L)$, $\Delta_{i_1}(\Gamma)$ and $\Delta_{i_1}(T)$ instead of $\Delta^{(3)}$ to describe situations where the site i_1 is a n.n. of i_0 or coincides with it. The definition of these matrices is clear from Fig.2 where broken lines denote new matrix elements weighted by $-\epsilon$ and double lines denote the element $[j_3, i_0] = 0$. The rest of elements coincide with those of Δ. Taking into account the left-right symmetry, we obtain the total contribution of configurations generating a Θ-graph :

$$N(\Theta) = -\lim_{\epsilon \to \infty} \Big\{ \sum_{i_1}{}' det\Delta_{i_1}(L) + \sum_{i_1}{}'' det\Delta_{i_1}(\Gamma) +$$
$$+ 2det\Delta_{j_4}(L) + 2det\Delta_{j_4}(\Gamma) + 2det\Delta_{j_4}(T) \Big\} / \epsilon^3 \qquad (10)$$

where the first sum runs over all lattice sites except j_2, j_4, i_0 and the second one except j_2, j_4, i_0, j_3.

Combining (10) with previous definitions, we obtain

$$N(C_0) = det\Delta^{(1)} + \lim_{\epsilon \to \infty} det\Delta^{(2)}/\epsilon + N(\Theta) \qquad (11)$$

The calculation of $N(C_0)/N$ is straightforward due to the formula $(det\Delta')/(det\Delta) = det(I - G\delta)$, where the matrix $G = \Delta^{-1}$ and the matrix $\delta = \Delta' - \Delta$. The non-zero elements of defect matrices in (10) and (11) occur only in four rows and columns. So, one needs to calculate merely 4×4 determinants, whose elements are given in terms of matrix elements of G. Summing over all positions of the site i_1 we get the quoted formula (1).

The developed technique may be applied to the evaluation of $P(3), P(4)$ and various correlation functions but the latter need a more elaborate consideration.

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References

[1] P.Bak, C.Tang, and K.Wiesenfeld, Phys.Rev.Lett. **59**, 381 (1987); Phys.Rev.A **38**, 364 (1988).

[2] D.Dhar, Phys.Rev.Lett.**64**,1613 (1990).

[3] S.N.Majumdar, D.Dhar, J.Phys.A**24**,L357 (1991).

[4] D.Dhar,S.N.Majumdar, J.Phys.A**23**,4333 (1990).

[5] Y.C.Zhang, Phys.Rev.Lett.**63**,470 (1989).

[6] A.Erzan and S.Sinha, Preprint ICTP,Trieste (1990).

[7] S.S.Manna, J.Stat.Phys.**59**,509 (1990).

[8] F.Harary, Seminar on Graph Theory,Nolt,N.Y. (1967).

[9] V.B.Priezzhev, Sov.Phys.Usp.**28**,1125 (1985).

[10] M.Hall,Combinatorial Theory,Blaisdell (1967).

Figure captions

Fig.1. The configuration of arrows responsible for $N(C_0)$.

Fig.2. Sites and bonds contributing to the definitions of defect matrices: (a) $\Delta_{i_1}(L)$; (b) $\Delta_{j_4}(T)$; (c) $\Delta_{i_1}(\Gamma)$.

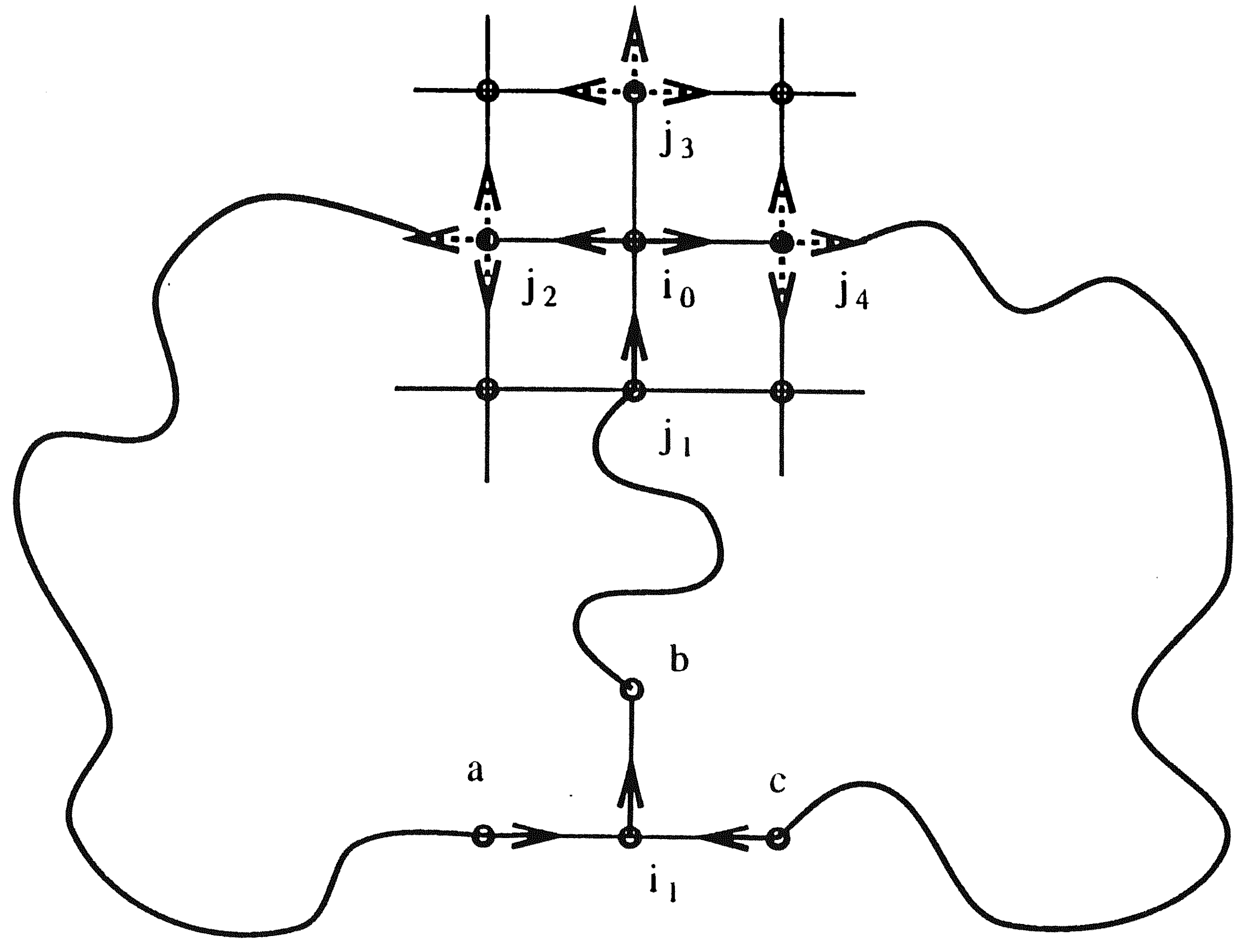
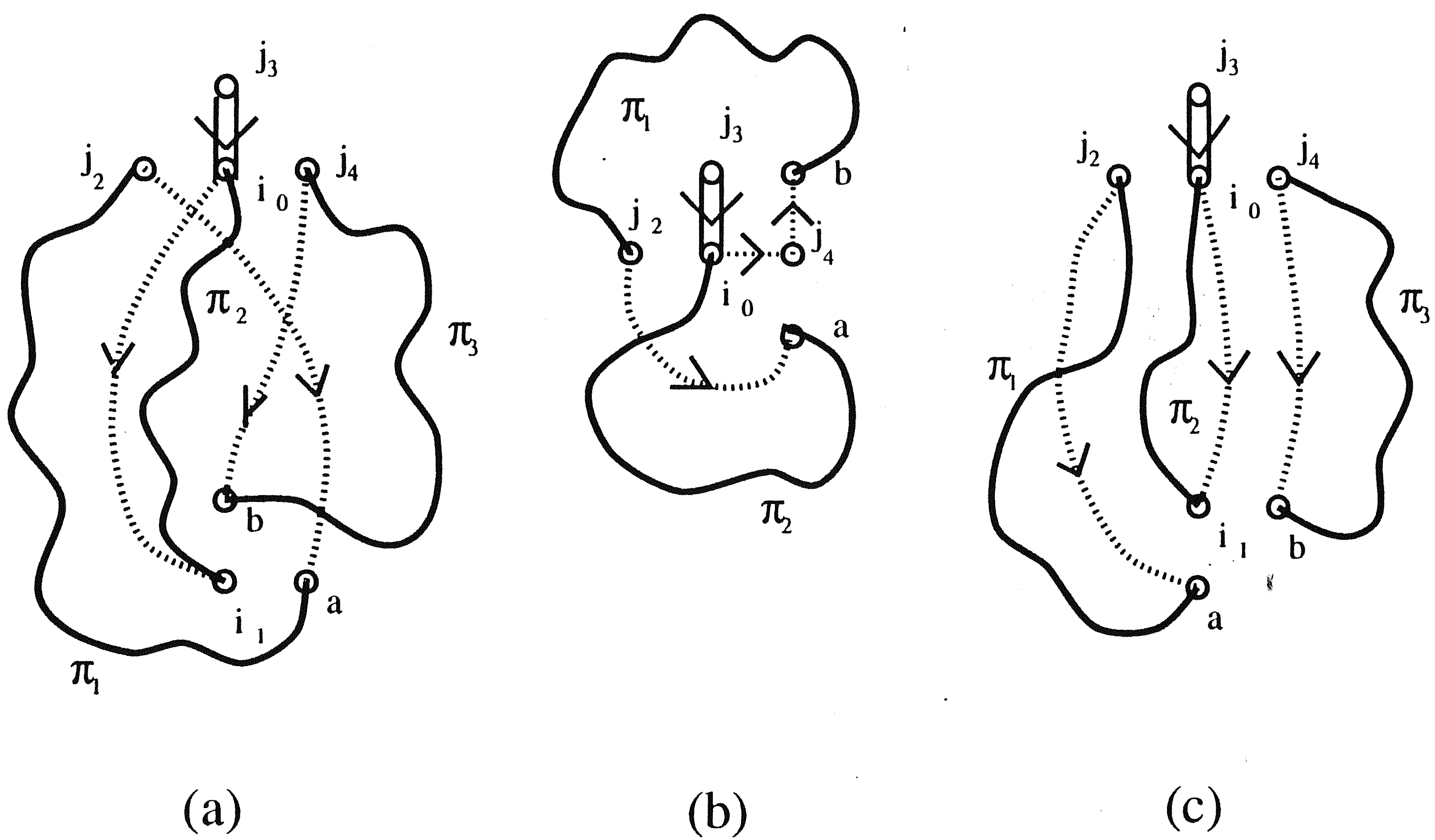


Fig. 1.



j_3
j_2
j_4
i_0
π_2
π_3
b
i_1
a
π_1
(a)
π_1
j_3
b
j_2
j_4
i_0
a
π_2
(b)
j_3
j_2
j_4
i_0
π_3
π_1
π_2
i_1
b
a
(c)

Fig.2.